The propagation of waves along and through a conducting layer of gas

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Two related questions concerning the transmission of electromagnetic waves are considered:

(i) The reflexion and transmission of plane waves at a perfectly conducting layer of gas in an otherwise non-conducting atmosphere, when there is a uniform external magnetic field perpendicular to the layer. Here the main result is that a layer of finite depth h is an almost perfect filter, being transparent to waves of frequency $n\pi A_0/h$ (A_0 = Alfvén velocity, n an integer).

(ii) The existence of plane surface waves for such a finite layer. There is always one such wave and, for certain ranges of frequency, two. The first becomes 'choked' at the filter frequencies, its velocity first tending to zero and then jumping to a finite value. The second chokes at the frequencies $n\pi A_0 a_0/h\sqrt{(a_0^2+A_0^2)}$ (a_0 = acoustic velocity).

1. Introduction

At large distances from a transmitter such as a monochromatic Hertzian dipole, situated in an unbounded atmosphere, the local electromagnetic field is essentially that of a plane wave. When a layer* of different material is also present, however, the field at points on the same side as the transmitter is modified by a reflected plane wave and, for a layer of suitable composition, by a surface wave. At points sufficiently close to the layer the latter provides by far the largest contribution to the field.

In electromagnetic theory, where the emphasis is on solid materials, surface waves are associated with the names of Zenneck and Sommerfeld [though their discovery was the result of mathematical errors, see e.g. Wait (1958)]. In the theory of elasticity their counterparts are known by the names of Rayleigh and Love. The main property of plane surface waves is that they propagate along the layer without change in waveform, their amplitudes decreasing exponentially with distance from the interface(s).

Here we consider a perfectly conducting layer in an otherwise non-conducting gas when there is a uniform external magnetic field perpendicular to the layer. Plane electromagnetic waves incident on the layer are in general completely reflected,† as in the case of a conducting solid. However, a layer of finite depth h

* Here the term 'layer' is also used for a half-space, i.e. a layer of infinite depth. In the following sections a distinction is made.

† This is meant in the energy sense. There is always a refracted wave, but it transports a negligible amount of energy in general.

is completely transparent to waves of frequency $n\pi A_0/h$; here A_0 is the Alfvén velocity and n is an integer. The layer is an almost perfect filter, and this applies whatever the polarization or angle of incidence (with the exception of certain grazing incidences).*

It appears that there are two types of surface wave for a layer of finite depth, corresponding to the two types of wave which can propagate through a conducting gas. The first can be excited by a transmitter of any frequency, though as a filter frequency is approached the wave becomes 'choked', its velocity tending rapidly to zero and then jumping to the velocity of light c directly the frequency is passed. The choking also marks a transition in the symmetry of the wave. For waves of the second type, the choking takes place at other frequencies, the velocity now jumping from zero to a_0, the acoustic speed in the non-conducting part of the gas. However, there are certain ranges of frequency for which they do not exist. There are no surface waves for a layer of infinite depth.

The discussion is based on the linearized form of the continuum equations. A finite coefficient of conductivity is retained in the basic equations since this provides the link between the waves inside and outside the layer and also avoids possible error in the boundary conditions. Simplifications arise from two sources: (i) the two types of wave motion can be treated separately—this would not be the case for any other orientation of the external field with respect to the layer; (ii) the propagation velocity of the incident electromagnetic wave is much greater than that of a refracted wave inside the layer (c large compared to a_0 and A_0), which means that the refracted waves cross the layer almost normally. It is essential, however, that a_0/c and A_0/c are not neglected indiscriminately.

2. Basic equations

We consider the motion of an electrically conducting inviscid gas in the absence of body forces and heat conduction. Further we shall assume that the material coefficients μ (permeability), ϵ (dielectric constant), and σ (conductivity) are constant. Then, as has been shown in a previous paper (Ludford 1959), the equations governing small perturbations of a given uniform state

$$\mathbf{E} = 0, \quad \mathbf{H} = \mathbf{H}_0, \quad \mathbf{v} = 0, \quad p = p_0, \quad \rho = \rho_0$$

are

$$\left.\begin{aligned} \mu\frac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl}\mathbf{E}, \quad \frac{1}{\sigma}\mathbf{J} = \mathbf{E}+\mu\mathbf{v}\times\mathbf{H}_0, \\ \rho_0\frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad}p+\mu\mathbf{J}\times\mathbf{H}_0, \\ \frac{\partial\rho}{\partial t}+\rho_0\operatorname{div}\mathbf{v} = 0, \quad \frac{\partial p}{\partial t}-a_0^2\frac{\partial\rho}{\partial t} = 0, \end{aligned}\right\} \tag{1}$$

where

$$\mathbf{J} = \operatorname{curl}\mathbf{H}-\epsilon\frac{\partial\mathbf{E}}{\partial t}.$$

In these equations $\mathbf{H}$, p, ρ denote deviations from $\mathbf{H}_0$, p_0, ρ_0, while $a_0^2 = (dp/d\rho)_0$, where this derivative is to be evaluated for fixed entropy.

* In some unpublished work, F. J. Fishman has found a similar result for the special case of normal incidence. This type of filtering action may have implications for propagation in the ionosphere.

The plane wave solutions may be determined as follows. Take the y-axis along $\mathbf{H}_0$ and let the x, y-plane contain the direction of propagation.* Then with all variables proportional to $\exp i(\omega t - \kappa x - \lambda y)$ and the same symbols used for factors, the equations divide into two sets, I and II, involving only E_1, E_2, H_3, v_3 for type I, and $E_3, H_1, H_2, v_1, v_2, p, \rho$, for type II. We omit the details (cf. paper cited above) and give only the results.

For waves of type I we have

$$E_1 = \frac{\kappa^2 - k^2}{\kappa\lambda} E_2 = \frac{\kappa^2 - k^2}{e\omega\lambda} H_3 = \frac{\rho_0}{H_0}(A_0^2 + i\eta\omega) v_3, \tag{2}$$

while κ, λ, and ω must satisfy the dispersion relation

$$\lambda^2 + (1-\delta)(\kappa^2 - k^2) = 0, \quad \delta = \frac{A_0^2 c^2}{(A_0^2 + i\eta\omega)(c^2 + i\eta\omega)}. \tag{3}$$

Here e is the 'complex dielectric constant' frequently used in the optics of absorbing media

$$e = \epsilon - \frac{i\sigma}{\omega} \quad \text{and} \quad k^2 = \mu e \omega^2 = \frac{\omega^2}{c^2} - \frac{i\omega}{\eta}, \tag{4}$$

while $c^2 = 1/\mu\epsilon$, $A_0^2 = \mu H_0^2/\rho_0$, and $\eta = 1/\mu\sigma$ (magnetic diffusivity).

For waves of type II we find $p = a_0^2 \rho$ and

$$E_3 = \frac{\mu\omega}{\lambda} H_1 = -\frac{\mu\omega}{\kappa} H_2 = -\frac{i\mu\omega H_0}{\eta(\kappa^2 + \lambda^2 - k^2)} v_1 = \frac{i\mu\omega H_0(\lambda^2 - \omega^2/a_0^2)}{\eta\kappa\lambda(\kappa^2 + \lambda^2 - k^2)} v_2$$
$$= \frac{iA_0^2(\lambda^2 - \omega^2/a_0^2)}{H_0\eta\kappa(\kappa^2 + \lambda^2 - k^2)} p, \tag{5}$$

while the dispersion relation becomes

$$\eta^2(\lambda^2 + \kappa^2 - k^2)(\lambda^2 + \nu\kappa^2 - \omega^2/a_0^2) = \nu A_0^2(\lambda^2 - \omega^2/a_0^2), \quad \nu = \frac{i\eta\omega}{A_0^2 + i\eta\omega}. \tag{6}$$

The corresponding results for a non-conducting fluid are obtained in the limit $\eta \to \infty$ $(\sigma \to 0)$. From equations (2) and (3) we find the well-known formulas

$$E_1 = -\frac{\lambda}{\kappa} E_2 = -\frac{\lambda}{\epsilon\omega} H_3, \quad v_3 = 0, \tag{7}$$

where $\kappa^2 + \lambda^2 = \omega^2/c^2$; similarly, equations (5) and (6) yield

$$E_3 = \frac{\mu\omega}{\lambda} H_1 = -\frac{\mu\omega}{\kappa} H_2, \quad v_1 = v_2 = p = 0, \tag{8}$$

if $\kappa^2 + \lambda^2 = \omega^2/c^2$, and

$$v_1 = \frac{\kappa}{\lambda} v_2 = \frac{\kappa}{\omega\rho_0} p, \quad E_3 = H_1 = H_2, \tag{9}$$

if $\kappa^2 + \lambda^2 = \omega^2/a_0^2$. Thus the waves of type I reduce to electromagnetic waves with $\mathbf{E}$ polarized in the plane of $\mathbf{H}_0$ and the direction of propagation, while those of type II reduce either to the perpendicularly polarized waves or acoustic waves.

* This is not the same choice of axes as in the paper cited above. However, it is more convenient for the present purposes.

For future reference we also list here the first two terms in the expansions of δ and ν in ascending powers of η:

$$\left.\begin{aligned} \delta &= 1 - i\eta\omega\left(\frac{1}{A_0^2}+\frac{1}{c^2}\right)+\ldots, \\ \nu &= \frac{i\eta\omega}{A_0^2}+\frac{\eta^2\omega^2}{A_0^4}+\ldots. \end{aligned}\right\} \tag{10}$$

These will hold when the fluid is a good conductor. The detailed form of the waves in this case has also been discussed in Ludford (1959).

3. Waves of type I. Semi-infinite conducting space

Suppose that the fluid fills the whole of space, but that for $y < 0$ its conductivity is zero, while for $y > 0$ it is non-zero (but constant). All other given quantities ($\mu, \epsilon, H_0, p_0, \rho_0, a_0$) are to remain unchanged on crossing the interface $y = 0$.

When an electromagnetic wave i (of type I) is incident on the interface from below, there will be a reflected wave r and a refracted or transmitted wave t (both of type I). The ω's and κ's of all three waves will be the same, but

$$\lambda_r = -\lambda_i, \quad \lambda_t = \sqrt{[(1-\delta)(k^2-\kappa^2)]}$$

[see equation (3)], where the root with positive real part is to be taken. Now the tangential components of the (total) electric and magnetic fields must be continuous at the interface. On expressing E_1 in terms of H_3 for each wave [see equations (2) and (7)], this leads to the conditions

$$H_i + H_r = H_t, \quad H_i - H_r = \theta H_t, \tag{11}$$

where the subscript 3 has been suppressed and

$$\theta = \frac{\epsilon(k^2-\kappa^2)}{e\lambda_i\lambda_t} = \frac{\epsilon\sqrt{(k^2-\kappa^2)}}{e\lambda_i\sqrt{(1-\delta)}}. \tag{12}$$

Thus we have

$$H_r = \frac{1-\theta}{1+\theta}H_i, \quad H_t = \frac{2}{1+\theta}H_i. \tag{13}$$

In particular, if the fluid is perfectly conducting above the interface,

$$\lambda_t = \frac{\omega}{A_0 c}\sqrt{(A_0^2+c^2)} \quad \text{and} \quad \theta = \frac{\omega A_0}{\lambda_i c\sqrt{(A_0^2+c^2)}}; \tag{14}$$

these limiting values as $\eta \to 0$ are obtained by using equations (4) and (10). The transmitted wave is an Alfvén wave which propagates in the $\mathbf{H}_0$-direction at a speed $A_0 c/\sqrt{(A_0^2+c^2)} \doteqdot A_0$, this value being independent of the incident wave. In terms of the angle of incidence ϕ_i we have

$$\theta = \frac{A_0}{\sqrt{(A_0^2+c^2)}}\sec\phi_i \doteqdot \frac{A_0}{c}\sec\phi_i, \tag{15}$$

which is an extremely small quantity except in the case of glancing incidence. Correspondingly we have

$$H_r = \tfrac{1}{2}H_t = H_i, \tag{16}$$

to a high degree of accuracy,* except for ϕ_i close to 90°. In the latter case, H_r decreases rapidly through zero to $-H_i$ and H_t decreases rapidly to zero, as $\phi_i \to 90°$. The corresponding range for the angle of refraction ϕ_t is extremely small, the largest possible value of $\tan\phi_t$ being given by

$$\max\frac{\kappa}{\lambda_t} = \frac{\omega}{c\lambda_t} = \frac{A_0}{\sqrt{(A_0^2+c^2)}} \doteqdot \frac{A_0}{c}.\dagger$$

For $\theta = 1$, i.e. the angle of incidence

$$\phi_i = \cos^{-1}\frac{A_0}{\sqrt{(A_0^2+c^2)}} \doteqdot \frac{\pi}{2} - \frac{A_0}{c}, \tag{17}$$

there is no reflected wave. The wave field is then the prototype for an alternating current in a 'fluid wire' (see, for example, Sommerfeld's treatment (1952, p. 160) of solid wires). Note the absence of a skin effect: λ_t is real and finite.

4. Waves of type I. Conducting layer

Now the conductivity of the fluid is assumed to be non-zero only within the layer $0 < y < h$. The refracted wave at $y = 0$ is also the incident wave at $y = h$ and will now be denoted by the subscript α; in addition, there will be a reflected wave β at the second interface and a transmitted wave t. The various values of λ are given by

$$\lambda_r = -\lambda_i, \quad \lambda_\alpha = -\lambda_\beta = \sqrt{[(1-\delta)(k^2-\kappa^2)]}, \quad \lambda_t = \lambda_i.$$

From the continuity of the tangential components of **E** and **H** at $y = 0$ we obtain the relations

$$H_i + H_r = H_\alpha + H_\beta, \quad H_i - H_r = \theta(H_\alpha - H_\beta), \tag{18a}$$

where θ is given by (12) (with λ_t replaced by λ_α in the first expression). Similar continuity requirements at $y = h$ yield the equations

$$\left.\begin{aligned} H_\alpha \exp(-i\lambda_\alpha h) + H_\beta \exp(-i\lambda_\beta h) &= H_t \exp(-i\lambda_t h), \\ H_\alpha \exp(-i\lambda_\alpha h) - H_\beta \exp(-i\lambda_\beta h) &= (1/\theta) H_t \exp(-i\lambda_t h). \end{aligned}\right\} \tag{18b}$$

Thus the reflected and transmitted waves are given by

$$\left.\begin{aligned} H_r &= \frac{(1/\theta - \theta)\, i \sin\lambda_\alpha h}{2\cos\lambda_\alpha h + i(1/\theta+\theta)\sin\lambda_\alpha h} H_i, \\ H_t &= \frac{2\exp(i\lambda_i h)}{2\cos\lambda_\alpha h + i(1/\theta+\theta)\sin\lambda_\alpha h} H_i. \end{aligned}\right\} \tag{19}$$

For a perfectly conducting layer, λ_α and θ are given by (14), or, in the case of θ, by (15). Since θ is very small, except for glancing incidence, it follows that the same is true of H_t, provided $\lambda_\alpha h$ is not close to a multiple of π. This last condition is violated when the frequency is near

$$\omega = \frac{A_0 c}{\sqrt{(A_0^2+c^2)}}\frac{n\pi}{h} \doteqdot \frac{n\pi A_0}{h} \quad \text{for some integer } n. \tag{20}$$

* This is essentially complete reflexion, since the electromagnetic waves transfer energy much faster than the Alfvén wave.

† If the incident wave were an Alfvén wave coming from above the interface, this would give the angle of incidence beyond which total reflexion occurs. Thus, effectively all such Alfvén waves are trapped.

The layer is therefore an almost perfect filter, allowing only waves with the frequencies (20) to be transmitted; to these waves it is transparent. For example, even if A_0 is as large as 10^5 cm/sec and angles of incidence up to 89° are admitted, the amplitude of H_t/H_i will have dropped from 1 to at most 1/40 by the time ω deviates by $\pi A_0/180h$ from its value (20). For normal incidence the decrease is to less than 1/2500 for the same deviation.

When $\theta = 1$ there is no reflected wave whatever the frequency. For the corresponding angle of incidence (17) the layer is transparent. For angles much closer to $\pi/2$ than (17), the filter action occurs again (θ large). Both of these last effects are almost certainly unattainable physically (for the above data ϕ_i is about $\frac{7}{10}$ of a second off 90°).

5. Waves of type II. Semi-infinite conducting space

Waves of the second type present more difficulty. For each given ω and κ there are two possible values of λ^2 satisfying (6), and hence two waves which may be excited. In the limit of zero conductivity these can be identified as electromagnetic and acoustic waves, the former carrying the variations in E_3, H_1, H_2 and the latter those in v_1, v_2, p (and ρ). In general, however, each wave bears the fluctuations of all six quantities and no distinction can be made.

Thus, when an electromagnetic wave (of type II) is incident on the single interface from below, there will be not only a reflected electromagnetic wave r but also a reflected acoustic wave a, and, in addition, two transmitted waves t and τ. All five waves will have the same ω's and κ's, while

$$\lambda_r = -\lambda_i, \quad \lambda_a = -\sqrt{(\omega^2/a_0^2-\kappa^2)},$$

and λ_t, λ_τ will be the roots of (6) which have positive real part.

The continuity of the tangential component of the (total) electric field implies that of the normal component of magnetic induction, see equations (5). To these must be added the continuity of the tangential component of the magnetic field, so that in all we obtain the two relations

$$\left.\begin{aligned} H_i + H_r &= H_t + \frac{1}{C_\tau} p_\tau, \\ H_i - H_r &= \frac{\lambda_i}{\lambda_t} H_t + \frac{\lambda_i}{\lambda_\tau}\frac{1}{C_\tau} p_\tau, \end{aligned}\right\} \tag{21a}$$

when E_3 is expressed in terms of H_1 or p and the subscript 1 omitted.* Here

$$C_\tau = \frac{\rho_0 \eta\omega\kappa(\kappa^2+\lambda_\tau^2-k^2)}{i H_0 \lambda_\tau(\lambda_\tau^2-\omega^2/a_0^2)}. \tag{22}$$

The remaining conditions are the continuity of pressure and of the normal component of velocity. When these two quantities are expressed in terms of H_1, for the wave t, and of p, for the waves a, and τ, we find

$$\left.\begin{aligned} p_a &= C_t H_t + p_\tau, \\ p_a &= \frac{\lambda_t}{\lambda_a} C_t H_t + \frac{\lambda_\tau}{\lambda_a} p_\tau, \end{aligned}\right\} \tag{21b}$$

where C_t is obtained from C_τ by replacing λ_τ with λ_t.

* The reason for using p instead of H_1 in the τ-wave will appear immediately.

Little is gained by writing down the solution of these four equations without first passing to the limit of infinite conductivity, and making certain approximations. In the limit we obtain

$$C = \frac{\rho_0 \omega^2 \kappa}{H_0 \lambda(\lambda^2 - \omega^2/a_0^2)}, \tag{23}$$

and the equation (6) for λ_t, λ_τ reduces, with the aid of equations (10), to

$$\left[\lambda^2 + \kappa^2 - \omega^2\left(\frac{1}{A_0^2} + \frac{1}{c^2}\right)\right]\left(\lambda^2 - \frac{\omega^2}{a_0^2}\right) = \frac{\omega^2 \kappa^2}{A_0^2}. \tag{24}$$

It is easily seen that this quadratic in λ^2 has positive roots when $\kappa \leqslant \omega/c$, so that λ_t and λ_τ are always real. We now use the fact that a_0/c and A_0/c are extremely small, which implies that κ and ω/c are small compared to ω/a_0 and ω/A_0. This yields the approximations*

$$\left.\begin{aligned} \lambda_t &= \frac{\omega}{A_0}\left[1 + \frac{A_0^2}{2\omega^2}\left(\frac{A_0^2}{a_0^2 - A_0^2}\kappa^2 + \frac{\omega^2}{c^2}\right)\right], \\ \lambda_\tau &= \frac{\omega}{a_0}\left[1 - \frac{a_0^4}{2\omega^2(a_0^2 - A_0^2)}\kappa^2\right], \end{aligned}\right\} \tag{25}$$

provided $a_0 \neq A_0$. To these may be added the approximation

$$\lambda_a = -\frac{\omega}{a_0}\left[1 - \frac{a_0^2}{2\omega^2}\kappa^2\right].$$

The expression for λ_τ shows us that the electromagnetic variables are of higher order than the dynamical in the τ-wave. From this follows the use of p_τ rather than H_τ above. In fact, the τ-wave is a modified acoustic wave and the t-wave a modified Alfvén wave [as may easily be seen from equations (5)], both of which move almost along the normal to the interface.†

With these approximations and with λ_i and κ expressed in terms of the angle of incidence ϕ_i, the solution becomes

$$\left.\begin{aligned} H_r &= \left(1 - 2\frac{A_0}{c}\cos\phi_i\right) H_i, & \frac{p_a}{\rho_0 a_0^2} &= -\frac{A_0}{a_0 + A_0}\frac{A_0}{c}\sin\phi_i \frac{H_i}{H_0}, \\ H_t &= 2\left(1 - \frac{A_0}{c}\cos\phi_i\right) H_i, & \frac{p_\tau}{\rho_0 a_0^2} &= -\frac{A_0}{a_0 - A_0}\frac{A_0}{c}\sin\phi_i \frac{H_i}{H_0}, \end{aligned}\right\} \tag{26}$$

where terms of the second order in a_0/c, A_0/c have been neglected. Comparison with equations (13) and (15) for waves of type I shows that $\sec\phi_i$ has been replaced by $\cos\phi_i$, as far as H_r and H_t are concerned. This means that equations (16) hold to a high degree of accuracy without qualification about the angle of incidence. The range of ϕ_t is A_0/c and that of ϕ_τ is a_0/c. There is always a reflected wave (this corresponds to the non-existence of magnetic principal waves on wires, see Sommerfeld (1952, p. 160) and the end of § 3). The acoustic waves a and τ are very weak, and in fact vanish for normal incidence.

* All we really need to know is that the second term in each bracket is of the second order. The case $a_0 = A_0$ is treated in Appendix A.

† The largest possible values of ϕ_t and ϕ_τ are A_0/c and a_0/c, respectively. If the incident wave were a t- or τ-wave coming from above the interface, the appropriate one of these would give the angle of incidence beyond which total reflexion (of electromagnetic effects) takes place. Thus, effectively all such waves are trapped.

6. Waves of type II. Conducting layer

The refracted waves at $y = 0$ are also incident waves at $y = h$ and will now be denoted by α, γ; in addition, there will be their reflexions β and δ, say, and two transmitted waves t and τ, the first electromagnetic and the second acoustic. The eight values of λ other than λ_i are determined by

$$\lambda_r = -\lambda_i, \quad \lambda_a = -\sqrt{[(\omega^2/a_0^2)-\kappa^2]}, \quad \lambda_\beta = -\lambda_\alpha,$$
$$\lambda_\delta = -\lambda_\gamma, \quad \lambda_t = \lambda_i, \quad \lambda_\tau = -\lambda_a,$$

where λ_α, λ_γ are the roots of equation (6) which have positive real part.

The continuity requirements at $y = 0$ lead to the equations

$$\left.\begin{aligned} H_i + H_r &= H_\alpha + H_\beta + \frac{1}{C_\gamma}(p_\gamma - p_\delta), \\ H_i - H_r &= \frac{\lambda_i}{\lambda_\alpha}(H_\alpha - H_\beta) + \frac{\lambda_i}{\lambda_\gamma}\frac{1}{C_\gamma}(p_\gamma + p_\delta), \\ p_a &= C_\alpha(H_\alpha - H_\beta) + p_\gamma + p_\delta, \\ p_a &= \frac{\lambda_\alpha}{\lambda_a}C_\alpha(H_\alpha + H_\beta) + \frac{\lambda_\gamma}{\lambda_a}(p_\gamma - p_\delta), \end{aligned}\right\} \tag{27a}$$

while those at $y = h$ give

$$\left.\begin{aligned} e_i H_t &= e_\alpha H_\alpha + \frac{1}{e_\alpha}H_\beta + \frac{1}{C_\gamma}\left(e_\gamma p_\gamma - \frac{1}{e_\gamma}p_\delta\right), \\ e_i H_t &= \frac{\lambda_i}{\lambda_\alpha}\left(e_\alpha H_\alpha - \frac{1}{e_\alpha}H_\beta\right) + \frac{\lambda_i}{\lambda_\gamma}\frac{1}{C_\gamma}\left(e_\gamma p_\gamma + \frac{1}{e_\gamma}p_\delta\right), \\ \frac{1}{e_a}p_\tau &= C_\alpha\left(e_\alpha H_\alpha - \frac{1}{e_\alpha}H_\beta\right) + e_\gamma p_\gamma + \frac{1}{e_\gamma}p_\delta, \\ -\frac{1}{e_a}p_\tau &= \frac{\lambda_\alpha}{\lambda_a}C_\alpha\left(e_\alpha H_\alpha + \frac{1}{e_\alpha}H_\beta\right) + \frac{\lambda_\gamma}{\lambda_a}\left(e_\gamma p_\gamma - \frac{1}{e_\gamma}p_\delta\right). \end{aligned}\right\} \tag{27b}$$

Here the C's are given by (22) with λ_r replaced by the appropriate λ. The e's stand for $\exp(-i\lambda h)$, with the suitable λ inserted, and give the changes in phase across the layer.

We pass immediately to the limit of infinite conductivity, and make the same approximations* as in the last section. The waves α, β are then seen to be modified Alfvén waves and γ, δ modified acoustic waves. The solution of the eight equations (27) for the two reflected and two transmitted waves turns out to be

$$\begin{aligned} \frac{H_r}{i\sin\lambda_\alpha h} &= \frac{(H_0 p_a/\rho_0 a_0^2)}{\{A_0[A_0\cos\lambda_\alpha h - a_0 i\sin\lambda_\alpha h - A_0\exp(-i\lambda_\gamma h)]/(a_0^2 - A_0^2)\}(A_0\sin\phi_i/c)} \\ &= \frac{(H_0 p_\tau/\rho_0 a_0^2)}{\{A_0\exp(-i\lambda_a h)[A_0 - \exp(-i\lambda_\gamma h)(A_0\cos\lambda_\alpha h + a_0 i\sin\lambda_\alpha h)]/(a_0^2 - A_0^2)\}(A_0\sin\phi_i/c)} \\ &= \frac{H_t}{2(A_0\cos\phi_i/c)\exp(i\lambda_i h)} = \frac{H_i}{i\sin\lambda_\alpha h + 2(A_0\cos\phi_i/c)\cos\lambda_\alpha h}, \end{aligned} \tag{28}$$

when terms of the second order are neglected.

* The case $a_0 = A_0$ is treated in Appendix A.

The values of H_r and H_t should be compared with equation (19) for waves of type I; $1/\theta$ has been replaced by $c/A_0 \cos\phi_i$, in accordance with what we found in the last section, and θ by zero. As a consequence the qualification about glancing incidence is absent* and the layer is an almost perfect filter for all angles of incidence, effectively transmitting only waves whose frequency satisfies $\sin\lambda_\alpha h = 0$, i.e.

$$\omega = \frac{n\pi A_0}{h}, \quad \text{with } n \text{ an integer.} \tag{29}$$

The acoustic waves are usually very weak, and in any case vanish for normal incidence. The incident wave is, in general, essentially reflected as an electromagnetic wave.

7. Surface waves of type I

We turn now to a related but quite different question, namely, the existence of wave systems, propagating without loss parallel to the layer,† in which the amplitudes of the waves outside the layer decrease exponentially with distance from the interface(s). The importance of such natural oscillations of the conducting region is that they tend to trap the energy radiated by an electromagnetic transmitter and transport it parallel to the layer [see Wait (1958) for a discussion of solid conductors].

Note that, in the case of the semi-infinite conducting space, we are not concerned with the total reflexion of waves coming from above the interface, though these give an exponentially damped electromagnetic wave. Such wave systems are produced by a source of energy deep within the conducting fluid. In the notation of § 3, we ask for a solution of (11) in which $H_i = 0$, the only condition on λ_r being that it is positive imaginary; ω is assumed given, while κ, henceforth assumed positive for definiteness, is to be determined. It is clear on physical grounds that there is no solution; since λ_t is real in equation (14), energy would be propagated away from the interface into the conducting region. This is confirmed by showing that $\theta = -1$ has no solution of the required kind.

For the finite conducting layer the situation is different. We ask for a solution of equations (18) with $H_i = 0$, the condition on $\lambda_r = -\lambda_t$ again being that it is positive imaginary. Physically the question makes sense. The Alfvén waves can combine to form a standing wave in y-direction, the energy transport taking place in the x-direction. In fact, this is what occurs, the wave system being symmetric about $y = \frac{1}{2}h$ for some values of h and antisymmetric for the rest.

On setting the determinant of equations (18) equal to zero [i.e. the denominator in either of equations (19)], we find the condition

$$\theta = \frac{\omega A_0}{\lambda_i c\sqrt{(A_0^2+c^2)}} = i\cot\tfrac{1}{2}\lambda_\alpha h, \quad -i\tan\tfrac{1}{2}\lambda_\alpha h,$$

where

$$\lambda_\alpha = \omega\frac{\sqrt{(A_0^2+c^2)}}{A_0 c}.$$

* In fact the filtering action improves as the angle of incidence increases, and the example for normal incidence in § 5 is now a conservative estimate for all angles of incidence.

† The conductivity is assumed infinite from now on.

Thus, remembering that $\lambda_i = \sqrt{[(\omega^2/c^2) - \kappa^2]}$, we have

$$\lambda_t = -i\frac{\omega A_0}{c\sqrt{(A_0^2+c^2)}}\tan\tfrac{1}{2}\lambda_\alpha h, \quad i\frac{\omega A_0}{c\sqrt{(A_0^2+c^2)}}\cot\tfrac{1}{2}\lambda_\alpha h,$$

$$\kappa^2 = \frac{\omega^2}{c^2}\left[1+\frac{A_0^2}{A_0^2+c^2}\tan^2\tfrac{1}{2}\lambda_\alpha h\right], \quad \frac{\omega^2}{c^2}\left[1+\frac{A_0^2}{A_0^2+c^2}\cot^2\tfrac{1}{2}\lambda_\alpha h\right].$$

The corresponding wave systems are

$$\frac{H_r}{2\cos\frac{1}{2}\lambda_\alpha h} = \frac{H_\alpha}{\exp(\frac{1}{2}i\lambda_\alpha h)} = \frac{H_\beta}{\exp(-\frac{1}{2}i\lambda_\alpha h)} = \frac{H_t}{2\cos\frac{1}{2}\lambda_\alpha h\exp(i\lambda_i h)},$$

and

$$\frac{H_r}{2i\sin\frac{1}{2}\lambda_\alpha h} = \frac{H_\alpha}{\exp(\frac{1}{2}i\lambda_\alpha h)} = -\frac{H_\beta}{\exp(-\frac{1}{2}i\lambda_\alpha h)} = -\frac{H_t}{2i\sin\frac{1}{2}\lambda_\alpha h\exp(i\lambda_i h)},$$

respectively, the first being symmetric and the second antisymmetric.

From the condition on λ_r it follows that the first system is acceptable only when $\tan\frac{1}{2}\lambda_\alpha h$ is positive and the second only when this quantity is negative. As ω increases, the change from one to the other takes place at the filter frequencies (20), the velocity ω/κ decreasing rapidly to zero and then jumping to c. Between two filter frequencies, the penetration depth $1/|\lambda_i|$ of the electromagnetic waves decreases from ∞ to 0 as ω increases.

8. Surface waves of type II

As in the case of waves of type I (and for a similar reason), there are no surface waves of type II when the conducting region is semi-infinite. This is shown formally in Appendix B.

We therefore pass to the conducting layer, and consider the determinant of the system (27) with $H_i = 0$. As before we require that $\lambda_r = -\lambda_t$ shall be positive imaginary, and, correspondingly, that λ_a shall be positive imaginary. This determinant splits into two factors, one corresponding to an antisymmetric wave-system in H and the other to a symmetric one. Thus κ satisfies one or other of the equations

$$\left.\begin{aligned}
&[(\omega^2/a_0^2)-\lambda_\alpha^2]\,(\lambda_\alpha\sin\tfrac{1}{2}\lambda_\alpha h - i\lambda_i\cos\tfrac{1}{2}\lambda_\alpha h)\,(\lambda_\gamma\sin\tfrac{1}{2}\lambda_\gamma h + i\lambda_a\cos\tfrac{1}{2}\lambda_\gamma h)\\
&\quad = [(\omega^2/a_0^2)-\lambda_\gamma^2]\,(\lambda_\gamma\sin\tfrac{1}{2}\lambda_\gamma h - i\lambda_i\cos\tfrac{1}{2}\lambda_\gamma h)\,(\lambda_\alpha\sin\tfrac{1}{2}\lambda_\alpha h + i\lambda_a\cos\tfrac{1}{2}\lambda_\alpha h),\\
&[(\omega^2/a_0^2)-\lambda_\alpha^2]\,(\lambda_\alpha\cos\tfrac{1}{2}\lambda_\alpha h + i\lambda_i\sin\tfrac{1}{2}\lambda_\alpha h)\,(\lambda_\gamma\cos\tfrac{1}{2}\lambda_\gamma h - i\lambda_a\sin\tfrac{1}{2}\lambda_\gamma h)\\
&\quad = [(\omega^2/a_0^2)-\lambda_\gamma^2]\,(\lambda_\gamma\cos\tfrac{1}{2}\lambda_\gamma h + i\lambda_i\sin\tfrac{1}{2}\lambda_\gamma h)\,(\lambda_\alpha\cos\tfrac{1}{2}\lambda_\alpha h - i\lambda_a\sin\tfrac{1}{2}\lambda_\alpha h),
\end{aligned}\right\} \quad (30)$$

where λ_α and λ_γ are determined as functions of κ by the equation (24).

The situation is not as simple as it was for waves of type I. We therefore consider two limiting cases, namely, (i) a_0 small, and (ii) A_0 small, and then (iii) the 'choking' phenomenon $\kappa = \infty$, for arbitrary a_0, A_0. The first corresponds to low temperature and the second to small applied magnetic field.

(i) When a_0 is small the roots of (24) are given by

$$\lambda_a = i\kappa, \quad \lambda_\gamma = \frac{\omega}{a_0},$$

correct to terms $O(1)$. Note that κ is no smaller than ω/a_0, since we require $\lambda_a^2 < 0$. The first of the relations (30) now reduces to

$$(\lambda_\alpha \sin \tfrac{1}{2}\lambda_\alpha h - i\lambda_i \cos \tfrac{1}{2}\lambda_\alpha h)\,(\lambda_\gamma \sin \tfrac{1}{2}\lambda_\gamma h + i\lambda_a \cos \tfrac{1}{2}\lambda_\gamma h) = 0,$$

where we may take $\lambda_i = -i\kappa$. Since the first factor is negative [it may be rewritten as $-\kappa(\sinh \frac{1}{2}\kappa h + \cosh \frac{1}{2}\kappa h)$], we are left with

$$i\lambda_a = -\lambda_\gamma \tan \tfrac{1}{2}\lambda_\gamma h = -\frac{\omega}{a_0}\tan\frac{1}{2}\frac{\omega h}{a_0},$$

which, because of the requirement on λ_a, can only be satisfied when $\tan \omega h/2a_0$ is positive. Then we have

$$\kappa = \frac{\omega}{a_0}\left|\sec\frac{1}{2}\frac{\omega h}{a_0}\right|, \quad \lambda_a = i\frac{\omega}{a_0}\tan\frac{1}{2}\frac{\omega h}{a_0}.$$

Similarly, the second of equations (30) can be satisfied only when $\tan \omega h/2a_0$ is negative and then

$$\kappa = \frac{\omega}{a_0}\left|\operatorname{cosec}\frac{1}{2}\frac{\omega h}{a_0}\right|, \quad \lambda_a = -i\frac{\omega}{a_0}\cot\frac{1}{2}\frac{\omega h}{a_0}.$$

As ω increases the change from one wave system to the other takes place at the values

$$\omega = \frac{n\pi a_0}{h}, \quad \text{with } n \text{ an integer}, \tag{31}$$

and not at the filter frequencies (29), as in the case of waves of type I. Between adjacent frequencies (31), the penetration depth $1/|\lambda_i|$ of the electromagnetic waves decreases from a_0/ω to zero, while that of the acoustic waves decreases from ∞ to 0.

(ii) We turn now to the case where A_0 is small, for which two possibilities for κ must be distinguished. If κ is small compared to ω/A_0, we find

$$\lambda_\alpha = i\sqrt{\left(\kappa^2 - \frac{\omega^2}{a_0^2}\right)}, \quad \lambda_\gamma = \frac{\omega}{A_0},$$

correct to terms $O(1)$, and the first of equations (30) reduces to

$$(\lambda_\gamma \sin \tfrac{1}{2}\lambda_\gamma h - i\lambda_i \cos \tfrac{1}{2}\lambda_\gamma h)\,(\sin \tfrac{1}{2}\lambda_\alpha h + i \cos \tfrac{1}{2}\lambda_\alpha h)\,\lambda_a = 0.$$

It follows that either $\lambda_a = 0$, i.e. $\kappa = \omega/a_0$, or else

$$i\lambda_i = \lambda_\gamma \tan \tfrac{1}{2}\lambda_\gamma h = \frac{\omega}{A_0}\tan\frac{1}{2}\frac{\omega h}{A_0},$$

i.e.
$$\kappa = \sqrt{\left(\frac{\omega^2}{c^2} + \frac{\omega^2}{A_0^2}\tan^2\frac{1}{2}\frac{\omega h}{A_0}\right)}. \tag{32}$$

Since λ_i must be negative imaginary and κ larger than ω/a_0, this last result holds for

$$\tan\frac{1}{2}\frac{\omega h}{A_0} \geqslant A_0\sqrt{\left(\frac{1}{a_0^2} - \frac{1}{c^2}\right)}. \tag{33}$$

Further approximation shows that the first result is valid (in the sense that λ_a is actually a small positive imaginary) when

$$\frac{1}{1 + 2/\omega h\sqrt{(1/a_0^2 - 1/c^2)}}A_0\sqrt{\left(\frac{1}{a_0^2} - \frac{1}{c^2}\right)} \leqslant \tan\frac{1}{2}\frac{\omega h}{A_0} \leqslant A_0\sqrt{\left(\frac{1}{a_0^2} - \frac{1}{c^2}\right)}.$$

The κ, ω-curve is continuous at the right end-point of such an interval; its approximate representation changes from $\kappa = \omega/a_0$ to equation (32), however.* On the other hand, for values of κ comparable with or greater than ω/A_0, a similar analysis shows that

$$\kappa = \frac{\omega}{A_0}\tan\frac{1}{2}\frac{\omega h}{A_0},$$

where once again the tangent must be positive. Clearly this is the completion of equation (32).

Similarly, the second of equations (30) yields equation (32) and the inequality (33) with the tangent replaced by the minus cotangent. The solution corresponding to $\lambda_a = 0$ above is the exact root $\lambda_\alpha = 0$, which must be rejected [$\kappa < \omega/a_0$, see equation (24)].

The pattern is now clear. As ω increases there is a change from one wave-system to the other at the filter frequencies (29). With the exception of the first interval, there is a portion at the lower end of each of the intervals between filter frequencies for which no wave system exists. In the intervals for which $\tan\omega h/2A_0$ is positive the penetration depth $1/|\lambda_i|$ of the electromagnetic waves decreases from $1/\omega\sqrt{(1/a_0^2 - 1/c^2)}$ to zero [at first like $1/\omega\sqrt{(1/a_0^2 - 1/c^2)}$] while that of the acoustic waves decreases from ∞ to 0 (at first remaining infinite). In the remaining intervals ($\tan\omega h/2A_0 < 0$) the total changes are the same, though the initial phases are absent.

(iii) Finally, we consider the 'choking' phenomenon $\kappa = \infty$, for arbitrary values of a_0 and A_0. When κ is large (in comparison with ω/a_0, ω/A_0), the roots of equation (24) are determined by

$$\lambda_\alpha = i\kappa, \quad \lambda_\gamma = \omega\sqrt{\left(\frac{1}{a_0^2}+\frac{1}{A_0^2}\right)},$$

correct to terms $O(1)$. Now the two relations (30) reduce to

$$\lambda_\gamma \sin\tfrac{1}{2}\lambda_\gamma h - \kappa\cos\tfrac{1}{2}\lambda_\gamma h = 0, \quad \text{i.e. } \kappa = \lambda_\gamma \tan\tfrac{1}{2}\lambda_\gamma h,$$

$$\lambda_\gamma \cos\tfrac{1}{2}\lambda_\gamma h + \kappa\sin\tfrac{1}{2}\lambda_\gamma h = 0, \quad \text{i.e. } \kappa = -\lambda_\gamma \cot\tfrac{1}{2}\lambda_\gamma h,$$

the first being valid for $\tan\frac{1}{2}\lambda h$ positive and the second for it negative. Thus, in fact, the choking takes place for the values

$$\omega = \frac{n\pi}{h}\frac{a_0 A_0}{\sqrt{(a_0^2+A_0^2)}}, \quad \text{with } n \text{ an integer,}$$

the previous approximations being in agreement with this value. In particular we see that, in contrast to waves of type I, it is fortuitous if a choking frequency is also a filter frequency.

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* The situation is similar to that of two half-asymptotes being the limit curve of a family of half-hyperbolas.

Appendix A

For $a_0 = A_0$ and $\kappa \neq 0$,* the approximations (25) are replaced by

$$\lambda_t = \frac{\omega}{a_0}\left[1+\frac{a_0\kappa}{2\omega}+\frac{a_0^2}{8\omega^2}\left(2\frac{\omega^2}{c^2}-3\kappa^2\right)\right],$$
$$\lambda_\tau = \frac{\omega}{a_0}\left[1-\frac{a_0\kappa}{2\omega}+\frac{a_0^2}{8\omega^2}\left(2\frac{\omega^2}{c^2}-3\kappa^2\right)\right],$$

where now it is necessary to know the second-order terms explicitly in order to determine the solution of the equations (21) correct to the first order. We find, in terms of the angle of incidence ϕ_i,

$$H_r = \left(1-2\frac{a_0}{c}\cos\phi_i\right)H_i, \quad \frac{p_a}{\rho_0 a_0^2} = -\frac{1}{2}\frac{a_0}{c}\sin\phi_i\frac{H_i}{H_0},$$
$$H_t = \left[1+\frac{a_0}{c}\left(\tfrac{1}{2}\operatorname{cosec}\phi_i-\cos\phi_i-\tfrac{1}{4}\sin\phi_i\right)\right]H_i,$$
$$\frac{p_\tau}{\rho_0 a_0^2} = -\left[1+\frac{a_0}{c}\left(\tfrac{1}{4}\sin\phi_i-\cos\phi_i\right)\right]\frac{H_i}{H_0}.$$

Thus the reflected waves are the same as for $A_0 \to a_0$ [see equations (26)], but the τ-wave is no longer weak, compensatory changes occurring in the t-wave.

Similarly, the solution of the system (27) for the layer becomes

$$\frac{H_r}{\tfrac{1}{2}(\overline{e_\alpha+e_\gamma}^2-4)+(a_0/c)\left[\tfrac{1}{2}(e_\alpha^2-e_\gamma^2)\operatorname{cosec}\phi_i+(\overline{e_\alpha+e_\gamma}^2-4)\cos\phi_i\right]}$$
$$= \frac{(H_0 p_a/\rho_0 a_0^2)}{(e_\gamma^2-e_\alpha^2)+(a_0/c)\left[(e_\gamma^2-e_\alpha^2)\cos\phi_i+(1-e_\alpha e_\gamma)\sin\phi_i-\tfrac{1}{2}(e_\alpha-e_\gamma)^2\operatorname{cosec}\phi_i\right]}$$
$$= \frac{e_i H_t}{-4(a_0/c)(e_\alpha+e_\gamma)\cos\phi_i}$$
$$= \frac{(1/e_a)(H_0 p_\tau/\rho_0 a_0^2)}{2(e_\gamma-e_\alpha)+(a_0/c)\left[6(e_\gamma-e_\alpha)\cos\phi_i+\tfrac{1}{2}(e_\alpha+e_\gamma)(1-e_\alpha e_\gamma)\sin\phi_i\right]}$$
$$= \frac{H_i}{\tfrac{1}{2}(\overline{e_\alpha+e_\gamma}^2-4)-(a_0/c)\left[8\cos\phi_i+\tfrac{1}{2}(e_\gamma^2-e_\alpha^2)\operatorname{cosec}\phi_i\right]},$$

so far as the transmitted and reflected waves are concerned. When these formulas are simplified by setting $e_\gamma = e_\alpha(1+i\omega h\sin\phi_i/c)$ the result is the formal limit of equations (28). Hence similar remarks to those at the end of § 6 apply here also.

Appendix B

The vanishing of the determinant of the system (21) with $H_i = 0$ yields the condition

$$\left(\frac{\omega^2}{a_0^2}-\lambda_t^2\right)(\lambda_t+\lambda_i)(\lambda_\tau-\lambda_a) = \left(\frac{\omega^2}{a_0^2}-\lambda_\tau^2\right)(\lambda_\tau+\lambda_i)(\lambda_t-\lambda_a), \tag{34}$$

* When $\phi_i = 0$ the results (26) and (28) hold even when $a_0 = A_0$. In fact in both cases $p_a = p_\tau = 0$ without approximation.

where the value of C has been taken from equation (23). Now λ_t and λ_a are required to be pure imaginaries, while for real $\kappa > \omega/a_0$ equation (24) has two real roots ($\pm\lambda_t$, say) and two purely imaginary roots ($\pm\lambda_r$). Since $\lambda_i \neq -\lambda_a$ for any κ, it follows [on taking real and imaginary parts of equation (34)] that

$$\left(\frac{\omega^2}{a_0^2}-\lambda_t^2\right)(\lambda_r-\lambda_a)=0, \quad \left(\frac{\omega^2}{a_0^2}-\lambda_r^2\right)(\lambda_r+\lambda_i)=0$$

must hold simultaneously. It is now easily checked that none of the four possibilities which these last conditions offer is compatible with λ_t and λ_r satisfying (24).

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